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Phase equivalent potentials, complex coordinates and supersymmetric quantum mechanics

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Abstract

Supersymmetric quantum mechanics may be used to construct reflectionless potentials and phase-equivalent potentials. The exactly solvable case of the λ sech^2 potential is used to show that for certain values of the strength λ the phase-equivalent singular potential arising from the elimination of all the bound states is identical to the original potential evaluated at a point shifted in the complex coordinate space. This equivalence has the consequence that certain general relations valid for reflectionless potentials and phase-equivalent potentials lead to hitherto unknown identities satisfied by the associated Legendre functions. This exactly solvable problem is used to demonstrate some aspects of scattering theory.

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1. Introduction

The connection between multi-soliton solutions of the Korteweg–deVries (KdV) equation and reflectionless potentials in non-relativistic quantum mechanics is well known (Scott *et al* 1973). For studying the bound states of quarks with definite angular momentum solutions of the non-relativistic Schrödinger equation in a confining potential have been considered. The s-state solution for the zero angular momentum case can be viewed as the odd states of a symmetric confining potential in the infinite space $-\infty \leqslant x \leqslant \infty$ (Thacker *et al* 1978, Quigg and Rossner 1981). Since confining potentials have no scattering states and have only bound states the confining potentials belong to the category of symmetric reflectionless potentials. Reflectionless potentials may also be constructed by starting from the free particle case and adding the bound states by the methods of supersymmetric quantum mechanics (SUSYQM) (Sukumar 1986). The two methods of constructing reflectionless potentials have been shown to be equivalent.

SUSYQM can also be used to start from a potential with a given spectrum and scattering phase shifts and eliminate the bound states to find a potential which is phase equivalent to the starting potential but supporting fewer bound states (Baye 1987a, Baye 1987b, Sukumar 1985). This construction leads to a definite relation between the two potentials. In this paper we consider the exactly solvable case of the $\lambda\,\mathrm{sech}^2$ potential which corresponds to the category of reflectionless potentials for special values of the strength λ. We consider this potential in r-space and its phase-equivalent partner. Various mathematical properties of the solutions in the two potentials are of interest. In section 2 of this paper we examine the relation between the solutions in two potentials. In section 3 we show that the general theories of relectionless potentials and phase-equivalent potentials lead to certain identities. The procedure is illustrated in section 4 by using a simple example. Section 5 contains a discussion of the main results of the paper. Units in which $\hbar = 1$ and the mass $\mu = \frac{1}{2}$ are used throughout this paper so that $\frac{\hbar^2}{2\mu} = 1$.

2. sech^2 and cosech^2 potentials and complex coordinates

We consider the Schrödinger equations for two potentials, one attractive and the other repulsive, of the form

$$V_d = -2n(2n+1)\cosh^{-2} r \tag{1}$$

$$V_s = +2n(2n+1)\sinh^{-2} r \tag{2}$$

for integer values of n and energy $E_j = -\gamma_j^2$ so that

$$\frac{\mathrm{d}^2\Psi_j}{\mathrm{d}r^2} = \left(-2n\,(2n+1)\cosh^{-2} r + \gamma_j^2\right)\Psi_j, \tag{3}$$

$$\frac{\mathrm{d}^2\Phi_j}{\mathrm{d}r^2} = \left(+2n\,(2n+1)\sinh^{-2} r + \gamma_j^2\right)\Phi_j. \tag{4}$$

Under the substitution $z = \tanh r$ equation (3) transforms to

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + 2n(2n+1) - \frac{\gamma_j^2}{1-z^2}\right]\Psi_j = 0 \tag{5}$$

which is the differential equation satisfied by the associated Legendre functions. Polynomial solutions arise for integer values of γ_j. The n bound-state energy eigenvalues of V_d and the corresponding normalized eigenfunctions satisfying vanishing boundary conditions at $z = 0$ and $z = 1$ are given by

$$\begin{aligned} &\gamma_j = (2j-1), \quad E_j = -\gamma_j^2, \quad j = 1, 2, \ldots, n \\ &\Psi_j = \alpha_j P_{2n}^{2j-1}(\tanh r), \qquad \alpha_j = \sqrt{\frac{2(2j-1)\,((2n+1-2j)!)}{(2n-1+2j)!}} \end{aligned} \tag{6}$$

with $j = n$ corresponding to the ground state and $j = 1$ to the highest lying bound state. The associated Legendre functions can be given in the explicit form (Abramowitz and Stegun 1965)

$$\begin{aligned} z < 1, \qquad & P_n^m(z) = (-)^m \frac{(1-z^2)^{\frac{m}{2}}}{2^n n!}\frac{\mathrm{d}^{n+m}}{\mathrm{d}z^{n+m}}(1-z^2)^n, \\ y > 1, \qquad & P_n^m(y) = +\frac{(y^2-1)^{\frac{m}{2}}}{2^n n!}\frac{\mathrm{d}^{n+m}}{\mathrm{d}y^{n+m}}(y^2-1)^n. \end{aligned} \tag{7}$$

V_s is a repulsive potential which supports no bound states. However it is still possible to find a solution to equation (4) at energies corresponding to the bound states of V_d. Under the substitution $y = \coth r$ equation (4) transforms to the same form as equation (5) with y replacing z and therefore the solutions in V_s at energies corresponding to the bound states of V_d may be given in the form

$$\gamma_j = (2n + 1 - 2j), \qquad \Phi_j = -\alpha_j P_{2n}^{2n+1-2j}(\coth r), \qquad j = 1, 2, \ldots, n. \tag{8}$$

The additional minus sign in the expression for Φ_j is placed to ensure that Φ_j and Ψ_j have the same asymptotic limit as $r \to \infty$.

Under the coordinate transformation $r \to r + \mathrm{i}\pi/2$

$$\begin{aligned} &\sinh r \to \mathrm{i}\cosh r, \qquad \cosh r \to \mathrm{i}\sinh r, \qquad \tanh r \to \coth r, \\ &\cosh^{-2} r \to -\sinh^{-2} r, \qquad V_d \to V_s \end{aligned} \tag{9}$$

which enables the identification

$$\Phi_j(r) \sim \Psi_j\left(r + \mathrm{i}\frac{\pi}{2}\right). \tag{10}$$

In fact the two functions in equation (10) are equal within a possible phase factor which may be chosen so that the unnormalizable functions $\Phi_j(r)$ have the same asymptotic limit as $\Psi_j(r + \mathrm{i}\pi/2)$ when $r \to \infty$.

Thus we have shown that the eigenfunctions at the same energy for the two potentials are simply related by a shift of the coordinate in the complex r-plane. We have considered the solutions in these special potentials because they play an active role in the discussion in the next section.

3. Reflectionless potentials and phase-equivalent potentials

A symmetric reflectionless potential in $-\infty \leqslant x \leqslant +\infty$ with binding energies $E_j = -\gamma_j^2$, $j = 1, 2, \ldots, N$ may be constructed by adding bound states to a free particle potential $V_0 = 0$ using a method for addition of bound states based on supersymmetric quantum mechanics (Sukumar 1986) and is given by

$$\begin{aligned} V(x) &= -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \operatorname{Det} M \\ M_{kj}(x) &= \frac{\gamma_j^{k-1}}{2}(\exp \gamma_j x + (-1)^{j+k} \exp -\gamma_j x), \qquad k, j = 1, 2, \ldots, N. \end{aligned} \tag{11}$$

It has been shown that it is possible to represent V in terms of the normalized eigenstates in the form

$$V(x) = -4\sum_{j=1}^{N} \gamma_j \Psi_j^2. \tag{12}$$

This potential is equivalent to the N soliton solution of the KdV at time $t = 0$ constructed by other methods (Scott *et al* 1973). If the γ_j are chosen to be integers in the sequence $[1, 2, \ldots, N]$ then the potential defined by equation (11) can be shown to be a $\mathrm{sech}^2 x$ potential (see the appendix for the proof), i.e,

$$V(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \operatorname{Det} M = -\frac{N(N+1)}{\cosh^2 x}. \tag{13}$$

The eigenstates of this potential are associated Legendre polynomials as shown by equation (6) in the last section. The representation of V in terms of normalized eigenstates in equation (11) and the variable $z = \tanh x$ may be used to obtain the relation

$$N(N+1)(1-z^2) = 4\sum_{m=1}^{N} m^2 \frac{(N-m)!}{(N+m)!}\left(P_N^m(z)\right)^2. \tag{14}$$

The equality identified in equation (14) seems to be a new result and is not listed in the usual texts on the properties of associated Legendre polynomials.

If N is chosen to be an even number $N = 2n$, the antisymmetric states of $V(x)$ may be viewed as the n eigenstates of a potential $V(r)$, $0 \leqslant r \leqslant \infty$. We now concentrate on the radial domain and choose $N = 2n$. If the γ_j values in equation (11) are in the integer sequence $[1, 2, \ldots, 2n]$ the potential arising from equation (11), after the change of variable $x \to r$, is the potential V_d defined in equation (1). As noted before the n bound states of V_d correspond to γ_j values in the odd integer sequence $[1, 3, \ldots, 2n-1]$. We next examine the scattering states of V_d. Using the free particle solution $\sin \kappa r$ at energy $E = \kappa^2$ and a general result connecting the eigenfunctions of supersymmetric partner potentials at the same energy (see the appendix) it is possible to represent the scattering state of $V(r)$ for the same positive energy E in the form

$$\Psi(\kappa, r) = \frac{\mathrm{Det}\, D(r)}{\mathrm{Det}\, M(r)}, \tag{15}$$

where the matrix M is of dimension $(N \times N)$ and the elements of the matrices M and D are given by

$$\begin{aligned}
&D_{kj}(r) = M_{kj}(r), \qquad k, j = 1, 2, \ldots, 2n\\
&D_{2n+1,j}(r) = M_{2n+1,j}(r), \qquad j = 1, 2, \ldots, 2n\\
&M_{kj}(r) = \frac{j^{k-1}}{2}(\exp jr + (-)^{j+k}\exp -jr)\\
&D_{k,2n+1}(r) = \frac{\mathrm{d}^{k-1}}{\mathrm{d}r^{k-1}}\sin \kappa r, \qquad k = 1, 2, \ldots, 2n+1.
\end{aligned} \tag{16}$$

The elements of the matrices M and D have a simple form in the limit $r \to \infty$ because the exponentially decaying parts of M_{kj} vanish in this limit. The exponentially growing parts scale and contribute a term to the determinant of M which cancels a corresponding term coming from the determinant of D. It may be shown that if a matrix A has elements $A_{kj} = \gamma_j^{k-1}$, $j, k = 1, 2, \ldots, N$ then

$$\mathrm{Det}\, A = \prod_{j=1}^{N-1}\prod_{k>j}^{N}(\gamma_k - \gamma_j). \tag{17}$$

Using this theorem repeatedly by setting $\gamma_{2n+1} = \pm \mathrm{i}\kappa$ the determinants of the matrices M and D may be evaluated in the $r \to \infty$ limit to give

$$Lt_{r\to\infty}\Psi(\kappa, r) = -\frac{\mathrm{i}}{2}(\exp \mathrm{i}\kappa r)\prod_{j=1}^{2n}(-j + \mathrm{i}\kappa) + \frac{\mathrm{i}}{2}(\exp -\mathrm{i}\kappa r)\prod_{j=1}^{2n}(-j - \mathrm{i}\kappa) \tag{18}$$

and hence the phase shift may be calculated to be

$$\delta = \frac{1}{2\mathrm{i}}\ln\left(\prod_{j=1}^{2n}\frac{+\mathrm{i}\kappa - j}{-\mathrm{i}\kappa - j}\right) = n\pi - \sum_{j=1}^{2n}\arctan\frac{\kappa}{j}. \tag{19}$$

It is clear that this phase-shift relation satisfies Levinson's theorem (Levinson 1949, Swan 1968) which states that the phase shift at zero energy must equal π multiplied by the number of bound states.

We next study the properties of the repulsive potential $V_s(r)$ in equation (2). Since V_s can be obtained from V_d by the mapping $r \to r + \mathrm{i}\pi/2$ equations (11) and (12) may be used to represent V_s in the form

$$\begin{aligned} V_s(r) &= -2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln \mathrm{Det}\,\tilde{M}, \\ \tilde{M}_{kj}(r) &= M_{kj}\left(r + \mathrm{i}\frac{\pi}{2}\right), \\ M_{kj}(r) &= \frac{j^{k-1}}{2}(\exp jr + (-)^{j+k}\exp -jr), \qquad k, j = 1, 2, \ldots, 2n. \end{aligned} \tag{20}$$

The scattering states of V_s for $E = \kappa^2$ may be given in the form

$$\begin{aligned} \Phi(\kappa, r) &= \frac{\mathrm{Det}\,\tilde{D}(r)}{\mathrm{Det}\,\tilde{M}(r)}, \\ \tilde{D}_{kj}(r) &= \tilde{M}_{kj}(r), & k, j = 1, 2, \ldots, 2n \\ \tilde{D}_{2n+1,j}(r) &= \tilde{M}_{2n+1,j}(r), & j = 1, 2, \ldots, 2n \\ \tilde{D}_{k,2n+1}(r) &= \frac{\mathrm{d}^{k-1}}{\mathrm{d}r^{k-1}}\sin\kappa r, & k = 1, 2, \ldots, 2n+1. \end{aligned} \tag{21}$$

The same reasoning as that used to go from equation (15) to equation (18) for deriving the asymptotic form of $\Psi(\kappa, r)$ may be used to find the asymptotic form of $\Phi(\kappa, r)$ in the limit $r \to \infty$. The additional factors arising in the evaluation of the determinants of $\tilde{M}$ and $\tilde{D}$ due to the transformation $r \to r + \mathrm{i}\pi/2$ are the same in the limit $r \to \infty$ and cancel each other. Hence it is possible to show that

$$Lt_{r\to\infty}\Phi(\kappa, r) = Lt_{r\to\infty}\Psi(\kappa, r) \tag{22}$$

leading to the result that V_d and V_s defined by equations (1) and (2) have phase shifts which are equal within integral multiples of π for all positive energies. V_d supports n bound states but V_s is a repulsive potential with no bound states. V_s has a repulsive singularity at the origin of the form $2n(2n + 1)/r^2$.

It is known that when the phase shifts are equal within integral multiples of π for all positive energies but the bound states are missing then the deep potential with bound states and the singular potential without any bound states may be related by supersymmetry and the difference between the potentials can be expressed as (Baye 1987a, Baye 1987b, Baye and Sparenberg 1994, Sukumar and Brink 2004)

$$\begin{aligned} V_s &= V_d - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln \mathrm{Det}\,F \\ F_{jk} &= \int_0^r \Psi_j(y)\Psi_k(y)\,\mathrm{d}y. \end{aligned} \tag{23}$$

The solutions in V_s at energies corresponding to the missing bound states of V_d may be found by solving

$$F_{jk}\Phi_k(r) = \Psi_j(r), \qquad Lt_{r\to\infty}\Phi_j(r) = Lt_{r\to\infty}\Psi_j(r). \tag{24}$$

It may also be shown that in terms of the solutions in the two potentials at the energies corresponding to the bound-state energies of the deep potential

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln \operatorname{Det} F = \sum_{j=1}^{n} \Psi_j(r)\Phi_j(r),$$
$$V_s - V_d = -2\frac{\mathrm{d}}{\mathrm{d}r}\sum_{j=1}^{n}\Psi_j(r)\Phi_j(r). \tag{25}$$

Using the explicit forms of V_d and V_s given in equations (1) and (2) equation (25) may be integrated from r to ∞ to give

$$\frac{n(2n+1)}{\sinh r\cosh r} = \sum_{j=1}^{n}\Psi_j(r)\Phi_j(r). \tag{26}$$

Using the expressions for the wavefunctions given in equations (6) and (8) and the variable $z = \tanh r$ we can establish the equality

$$-\sum_{m=1,3,..}^{2n-1} P_{2n}^{m}(z)\,P_{2n}^{m}\left(\frac{1}{z}\right)2m\frac{(2n-m)!}{(2n+m)!} = n(2n+1)\frac{1-z^2}{z}. \tag{27}$$

This is another new identity not listed in the usual texts on the properties of associated Legendre functions.

4. A simple example

The results in equations (14) and (26) may be illustrated by considering the case $N = 2$, $n = 1$. Using

$$P_2^1(z) = -3z\sqrt{1-z^2}, \qquad P_2^2(z) = 3(1-z^2) \tag{28}$$

it can be checked that

$$6(1-z^2) = 4\left(1^2\frac{1!}{3!}\left(P_2^1(z)\right)^2 + 2^2\frac{0!}{4!}\left(P_2^2(z)\right)^2\right) \tag{29}$$

verifying the equality in equation (14) for the case $N = 2$.

Using

$$\Psi_1(r) = -\sqrt{3}\frac{\tanh r}{\cosh r}, \qquad F_{11} = \int_0^r \Psi_1^2(y)\,\mathrm{d}y = \tanh^3 r$$
$$F_{11}\Phi_1 = \Psi_1, \qquad \Phi_1(r) = -\sqrt{3}\frac{\coth r}{\sinh r} = i\Psi_1\left(r + \mathrm{i}\frac{\pi}{2}\right) \tag{30}$$

it can be seen that

$$\Psi_1(r)\Phi_1(r) = +\frac{3}{\sinh r\cosh r} \tag{31}$$

thereby verifying equation (26) for the case $n = 1$.

The scattering state of the potential in equation (1) for the case $n = 1$ for positive energy $E = -\kappa^2$ constructed using equations (12), (15) and (16) with $\gamma_1 = 1$ and $\gamma_2 = 2$ is given by

$$\Psi(\kappa, r) = \left(\kappa^2 - 2 + \frac{3}{\cosh^2 r}\right)\sin\kappa r + 3\kappa\tanh r\cos\kappa r \tag{32}$$

leading to the phase shift

$$\delta = \pi - \arctan\kappa - \arctan\frac{\kappa}{2}. \tag{33}$$

The scattering state of the potential in equation (2) for the case $n = 1$ constructed using equations (20) and (21) is given by

$$\Phi(\kappa, r) = \left(\kappa^2 - 2 - \frac{3}{\sinh^2 r}\right) \sin\kappa r + 3\kappa \coth r \cos\kappa r \tag{34}$$

leading to the phase shift

$$\tilde{\delta} = -\arctan\kappa - \arctan\frac{\kappa}{2}. \tag{35}$$

It was noted earlier that V_s is the phase-equivalent singular potential arising from V_d by the elimination of the bound states of V_d. Equations (33) and (35) show that the zero energy phase shifts satisfy Levinson's theorem (Swan 1968) as applied to the case of a missing bound state and that for all positive energies the phase shifts differ by π. Thus the phase shifts of the two potentials are equal within a multiple of π when $n = 1$.

From the general theory for the construction of singular potentials by the elimination of bound states it may be shown that the wavefunctions in the phase-equivalent deep and singular potentials are related by

$$\Phi(\kappa, r) = \Psi(\kappa, r) - \sum_{j=1}^{n} \Phi_j(r) \int_0^r \Psi_j(y)\Psi(\kappa, y)\, \mathrm{d}y \tag{36}$$

which may be simplified by using the Wronskian between $\Psi_j(r)$ and $\Psi(\kappa, r)$ to the form

$$\Phi(\kappa, r) = \Psi(\kappa, r) + \sum_{j=1}^{n} \frac{\Phi_j(r)}{\gamma_j^2 + \kappa^2} \left(\Psi_j(r)\frac{\mathrm{d}}{\mathrm{d}r}\Psi(\kappa, r) - \Psi(\kappa, r)\frac{\mathrm{d}}{\mathrm{d}r}\Psi_j(r)\right). \tag{37}$$

It may be verified that equations (32) and (34) satisfy equation (37).

In scattering theory the phase shift may be expressed in terms of the scattering wavefunction normalized to the asymptotic form $Lt_{r\to\infty}\Psi(\kappa, r) = \sin(\kappa r + \delta)$ by the exact expression (Messiah 1958)

$$\sin\delta = -\frac{1}{k}\int_0^\infty V(r) \sin\kappa r\, \Psi(\kappa, r)\, \mathrm{d}r. \tag{38}$$

For the case $n = 1$ use of the scattering wavefunction in equation (32) with appropriate changes to account for the different normalization leads to the integral relation

$$1 = +\frac{2}{\kappa^2}\int_0^\infty \frac{\sin\kappa r}{\cosh^2 r}\left(\left(\kappa^2 - 2 + \frac{3}{\cosh^2 r}\right)\sin\kappa r + 3\kappa \tanh r \cos\kappa r\right) \tag{39}$$

which can be verified by explicit evaluation of the integral.

The Born approximation limit of the phase shift for the potential $V_d = -6\cosh^{-2} r$ obtained by replacing $\Psi(\kappa, r)$ by $\sin\kappa r$ in equation (38) is

$$\sin\delta = \frac{6}{\kappa}\int_0^\infty \frac{\sin^2\kappa r}{\cosh^2 r}\mathrm{d}r = \frac{3}{k}\left(1 - \frac{\kappa\pi}{\sinh\kappa\pi}\right) \tag{40}$$

leading to

$$Lt_{\kappa\to\infty}\sin\delta = \frac{3}{\kappa}, \qquad Lt_{\kappa\to\infty}\delta = \frac{3}{\kappa} \tag{41}$$

which is in agreement with the limiting value of the phase shift obtained from equation (33) as given by

$$Lt_{\kappa\to\infty}\delta = \pi - \left(\frac{\pi}{2} - \frac{1}{\kappa}\right) - \left(\frac{\pi}{2} - \frac{2}{\kappa}\right) = \frac{3}{\kappa}. \tag{42}$$

5. Discussion

In this paper we have considered attractive sech^2 potentials of strength $\lambda = 2n(2n+1)$ for integer values of n. In the infinite space $-\infty \leqslant x \leqslant \infty$ this is a symmetric reflectionless potential. In the semi-infinite space $0 \leqslant r \leqslant \infty$ this potential has a definite phase shift for positive energies. By eliminating all the bound states of this potential it is possible to find a phase-equivalent potential which is a singular repulsive cosech^2 potential of strength $\lambda = 2n(2n+1)$. The eigenstates of these two potentials have been shown to be related by a shift of coordinate in the complex plane. If the strength $\lambda \neq 2n(2n+1)$ then the potential in the x-space is not of the reflectionless category. Furthermore the phase-equivalent singular potential generated by eliminating the bound states by the SUSY procedure is not simple and is not of the form of a cosech^2 potential. Hence the sech^2 potential of strength $\lambda = 2n(2n+1)$ belongs to a special category and we have shown that the solutions in this potential and its phase-equivalent partner lead to the identities given in equations (14) and (27). We have shown that the exactly solvable example considered in this paper enables a direct demonstration of the exactness of the general expression for the phase shift (equation (38)) in scattering theory. The relation between potentials, their phase-equivalent partners and transformations in complex coordinate space is an interesting subject which merits further study. Equation (12) which expresses the reflectionless potentials in terms of their eigenstates and equation (25) which expresses the difference between a potential with bound states and its phase-equivalent partner with fewer bound states in terms of the solutions in the two potentials at the energies of the eliminated bound states encapsulate a nonlinear structure which is quite general and not restricted to simple exactly solvable models.

Acknowledgments

I thank a referee for pointing out that equation (14) is a special case of an identity listed in the book Integrals and Series (volume 3) authored by Prudnikov, Brychkov and Marichov (1990).

Appendix

Supersymmetric quantum mechanics (Sukumar 1985)) may be used to remove or add bound states to a given potential. If we start from the potential

$$V_N(x) = -\frac{N(N+1)}{\cosh^2 x} \tag{A.1}$$

then it may be verified that

$$\xi_N(x) \sim \cosh^{-N} x \tag{A.2}$$

solves the Schrödinger equation for the potential V_N with the energy eigenvalue $E_N = -N^2$. Since ξ_N is a nodeless nomalizable function it must be the ground state of V_N. The normalized ground-state eigenfunction normalized to unity in the interval $[-\infty, +\infty]$ (Gradshteyn and Ryzhik 1965) is

$$\Psi_N(x) = \alpha_N \cosh^{-N} x, \qquad \alpha_N = \sqrt{\frac{(2N-1)!!}{2^N(N-1)!}}. \tag{A.3}$$

The supersymmetric partner to V_N which has the same spectrum as V_N except for missing the ground state at E_N is given by

$$\tilde{V}_N(x) = V_N(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \xi_N(x) = -\frac{(N-1)N}{\cosh^2 x}. \tag{A.4}$$

The normalized eigenfunctions Ψ of V_N and the normalized eigenfunctions Φ of $\tilde{V}_N$ for all energies $E \neq E_N$ are related by

$$\begin{aligned}\Phi(E,x) &= \frac{1}{\sqrt{(E-E_N)}} A_N^- \Psi(E,x), \qquad \Psi(E,x) = \frac{1}{\sqrt{(E-E_N)}} A_N^+ \Phi(E,x) \\ A_N^{\pm} &= \pm\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\xi_N}\frac{\mathrm{d}\xi_N}{\mathrm{d}x}.\end{aligned} \tag{A.5}$$

These intertwining relations are also valid for the scattering states for energy $E = \kappa^2$ and may also be given in the form

$$\begin{aligned}\Phi(\kappa,x) &= \frac{\det D(x)}{\Psi_N(x)} \\ D_{11} &= \Psi_N(x), \qquad D_{12} = \Psi(\kappa,x), \qquad D_{21} = \frac{\mathrm{d}}{\mathrm{d}x}\Psi_N(x), \qquad D_{22} = \frac{\mathrm{d}}{\mathrm{d}x}\Psi(\kappa,x).\end{aligned} \tag{A.6}$$

It is clear from equations (A.1) and (A.4) that $\tilde{V}_N$ may be obtained from V_N by the transformation $N \to (N-1)$. Hence it may be concluded that the ground-state energy of V_{N-1} is $E_{N-1} = -(N-1)^2$ with the ground-state eigenfunction

$$\xi_{N-1}(x) \sim \cosh^{-(N-1)}(x). \tag{A.7}$$

Using the intertwining operator A_N^+ in equation (A.5) the first excited state of V_N may be found in the form

$$\Psi_{N-1} \sim \frac{1}{\Psi_N}\frac{\mathrm{d}}{\mathrm{d}x}(\Psi_N \xi_{N-1}) \sim \cosh^N x \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cosh x}\right)\cosh^{2-2N} x. \tag{A.8}$$

This procedure may be iterated N times to show that the eigenvalue spectrum of V_N is given by $E_j = -j^2$, $j = 1, 2, \ldots, N$ and that the potential after N iterations is

$$\begin{aligned}V_0 &= V_N - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\xi \\ \xi &= \prod_{j=1}^{N}\xi_j = (\cosh x)^{-\frac{N(N+1)}{2}}\end{aligned} \tag{A.9}$$

which when evaluated gives $V_0 = 0$. By iteration of equation (A.8) the eigenfunctions Ψ_{N-j} of V_N with eigenvalues $E_{N-j} = -(N-j)^2$, $j = 0, 1, 2, \ldots, N-1$ may be given in the form

$$\begin{aligned}\Psi_{N-j} &\sim \frac{1}{\xi_N} A_N^+ A_{N-1}^+ \cdots A_{N+1-j}^+ \xi_{N-j} \\ &\sim \cosh^N x \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cosh x}\right)^j \cosh^{2j-2N} x, \qquad j = 0, 1, \ldots, N-1.\end{aligned} \tag{A.10}$$

The eigenstates may be normalized by including the energy denominators arising from factors similar to the ones appearing in equation (A.5) and the normalization factor α_{N-j} which can be found using equation (A.3). The normalized eigenstates are

$$\Psi_{N-j} = (2N-2j-1)!!\sqrt{\frac{(N-j)}{(2N-j)!j!}}\cosh^N x\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cosh x}\right)^J \cosh^{2j-2N} x. \tag{A.11}$$

Since the Schrödinger equation for V_N may be shown to be identical to the differential equation for the associated Legendre functions in terms of the variable $z = \tanh x$ the eigenfunctions are associated Legendre polynomials in z and the normalized eigenstates may be given in the form

$$\Psi_{N-j}(x) = \sqrt{\frac{j!}{(2N-j)!}(N-j)} P_N^{N-j}(\tanh x), \qquad j = 0, 1, 2, \ldots, N-1. \tag{A.12}$$

Comparison of equations (A.11) and (A.12) provides a new representation of the associated Legendre polynomials given by

$$P_N^{N-j}(\tanh x) = \frac{(2N-2j-1)!!}{j!}\cosh^N x \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cosh x}\right)^J \cosh^{2j-2N} x. \tag{A.13}$$

The above procedure may be reversed to start from the free particle potential $V_0 = 0$ and SUSYQM may be used to add the bound states at $E_j = -j^2$, $j = 1, 2, \ldots, N$. It has been shown (Sukumar 1986) that

$$\begin{aligned} V_N &= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \mathrm{Det}\, M \\ M_{kj}(x) &= \frac{\gamma_j^{k-1}}{2}(\exp\gamma_j x + (-1)^{j+k}\exp -\gamma_j x), \qquad k, j = 1, 2, \ldots, N \\ \gamma_j &= j. \end{aligned} \tag{A.14}$$

Comparison of the two expressions relating the potentials viewed as addition of N bound states or the removal of N bound states leads to the relation

$$\ln \mathrm{Det}\, M = -\ln\xi + \alpha x + \beta, \tag{A.15}$$

where α and β are constants which may be determined by examining the expressions in the limit $x \to \infty$:

$$\begin{aligned} Lt_{x\to\infty}\xi &= 2^{\frac{N(N+1)}{2}}\exp -\frac{N(N+1)}{2}x \\ Lt_{x\to\infty}\mathrm{Det}\, M &= 2^{-N}\exp\frac{N(N+1)}{2}x\,\mathrm{Det}\, A \\ A_{kj} &= j^{k-1}, \qquad k, j = 1, 2, \ldots, N. \end{aligned} \tag{A.16}$$

The determinant of the matrix A whose elements are powers of integers may be evaluated using equation (17) and by comparison of the two limits it can be shown that

$$\alpha = 0, \qquad \exp\beta = 2^{\frac{N(N-1)}{2}}\prod_{j=1}^{N-1} j!. \tag{A.17}$$

Thus it can be established that

$$\mathrm{Det}\, M = 2^{\frac{N(N-1)}{2}}(\cosh x)^{\frac{N(N+1)}{2}}\prod_{j=1}^{N-1} j! \tag{A.18}$$

and the resulting potential after the addition of N bound states at $E_j = -j^2$, $j = 1, 2, \ldots, N$, is

$$V_N = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \mathrm{Det}\, M = -\frac{N(N+1)}{\cosh^2 x}. \tag{A.19}$$

By generalizing the intertwining relations in equations (A.5) and (A.6) for the case of addition of N bound states it may be established that the scattering states of V_N for energy $E = \kappa^2$ may be related to the scattering solutions $\Phi(\kappa, x)$ in V_0 by

$$\begin{aligned} \Psi(\kappa, x) &= \frac{\mathrm{Det}\, D(x)}{\mathrm{Det}\, M(x)} \\ D_{kj} &= M_{kj}(x), \qquad && k, j = 1, 2, \ldots, N \\ D_{N+1,j} &= M_{N+1,j}(x), && j = 1, 2, \ldots, N \\ D_{k,N+1} &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1}\Phi(\kappa, x), && k = 1, 2, \ldots, N+1. \end{aligned} \tag{A.20}$$

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover) p 332
Baye D 1987a *Phys. Rev. Let* **58** 2738
Baye D 1987b *J. Phys. A: Math. Gen.* **20** 5529
Baye D and Sparenberg J M 1994 *Phys. Rev. Let.* **73** 2789
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic) p 344
Levinson N 1949 *Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd* **25** 9
Messiah A 1958 *Quantum Mechanics* (New York: Wiley) p 405
Prudnikov A P, Brychkov Yu A and Marichov O I 1990 *Integrals and Series* vol 3 (London: Gordon and Breach) p 387
Quigg C and Rosner J L 1981 *Phys. Rev.* D **23** 2625
Scott A C, Chu F Y E and Mclaughlin D W 1973 *Proc. IEEE* **61** 1443
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917, 2937
Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2297
Sukumar C V and Brink D M 2004 *J. Phys. A: Math. Gen.* **37** 5689
Swan P 1968 *Ann. Phys.* **48** 455
Thacker H B, Quigg C and Rosner J L 1978 *Phys. Rev.* D **18** 274, 287